CHAPTER 12

Vector Algebra

Important Definitions

- **Representation of Vectors:** A vector $\vec{a}$ is represented by the directed line segment $\overrightarrow{AB}$. The magnitude of the vector $\vec{a}$ is equal to $\overrightarrow{AB}$, and the direction of the vector $\vec{a}$ is along the line from A to B.
- **Scalar Quantity:** A quantity that has only magnitude and is not related to any direction is called a scalar quantity.
- **Vector Quantity:** A quantity that has magnitude and also a direction in space is called a vector quantity.
- **Null Vector or Zero Vector:** If the initial and terminal points of a vector coincide, then it is called a zero vector. It is denoted by $\vec{0}$ or O. Its magnitude is zero and direction indeterminate.
- **Unit Vector:** A vector whose magnitude is of unit length along my vector $\vec{a}$ is called a unit vector in the direction of $\vec{a}$ and is denoted by $\hat{a}$
- **Equal Vector:** Two non-zero vectors are said to be equal vectors if their magnitude is equal and directions are the same.
- **Collinear Vector:** Two or more non-zero vectors are said to be collinear vectors if they are parallel to the same line.
- **Like and Unlike Vector:** Collinear vectors having the same direction are known as like vectors, while those having opposite directions are known as, unlike vectors.
- **Coplanar Vector:** Two or more non-zero vectors are said to be coplanar vectors if these are parallel to the same plane.
- **Localised Vector and Free Vector:** A vector drawn parallel to a given vector through a specified point as the initial point, is known as a localised vector. If the initial point of a vector is not specified, it is said to be a free vector.
- **Position Vector:** Let O be the origin and A be a point such that $\overrightarrow{OA} = \vec{a}$, then we say that the position vector of A is $\vec{a}$.

Negative of a Vector

- Let $\overrightarrow{AB}$ be a vector directed from A to B. then $-\overrightarrow{AB}$ is a vector which would be directed from B to A.

Coinitial Vectors

- Two vectors are said to be coinitial vectors if both the vectors have the same initial points.

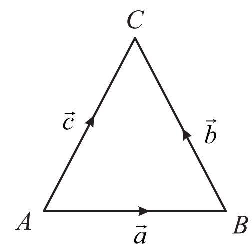
Co-terminal Vectors

- Two vectors are said to be Co-terminal vectors if both the vectors have the same terminating point.

Algebra of vectors

Addition of Vectors

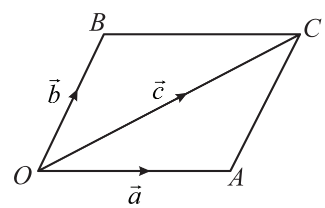
Triangle Law



Result: $\vec{a}+\vec{b}=\vec{c}$ or $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Converse of triangle law is also true.

Parallelogram Law



Result: $\vec{a}+\vec{b}=\vec{c}$ or $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$

Properties of vector addition:

(*i*) $\vec{a}\quad\vec{b}\quad\vec{b}\quad\vec{a}$ (commutative)

(*ii*) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associative)

(*iii*) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$

(*iv*) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

(*v*) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$

(*vi*) $|\vec{a}-\vec{b}|\ge||\vec{a}|-|\vec{b}||$

Multiplication of Vector by Scalars

If $\vec{a}$ and $\vec{b}$ are vectors & m, n are scalars, then

(*i*) $m(\vec{a})=(\vec{a})m=m\vec{a}$

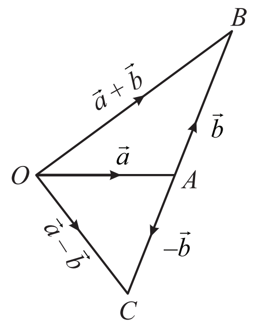
(*ii*) $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

(*iii*) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

(*iv*) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

Subtraction of Vectors

In the given diagram $\vec{a}$ and $\vec{b}$ are represented by $\overline{OA}$ and $\overline{AB}$. We extend the line AB in opposite direction upto C, where $AB = AC$. The line segment $\overline{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overline{OC}$ is $\vec{a}+(-\vec{b})$. i.e., denotes the vector $\vec{a}-\vec{b}$.



Note:

(*i*) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

(*ii*) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(*iii*) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtaction of vectors does not obey the associative law.

Important Properties and Formulae

- If $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ then $\vec{r_1}+\vec{r_2}=(x_1+x_2)\hat{i}+(y_1+y_2)\hat{j}+(z_1+z_2)\hat{k}$ and $\vec{r_1}=\vec{r_2}$ $\Leftrightarrow x_1=x_2,\ y_1=y_2,\ z_1=z_2$.
- $\vec{a}$ and $\vec{b}$ are parallel or collinear if $\vec{a}=m\vec{b}$ and only if for some non-zero scalar m.
- $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$ or $\vec{a}=|\vec{a}|\,\hat{a}$
- $\vec{r},\vec{a},\vec{b}$ are coplanar if and only if $\vec{r}=x\vec{a}+y\vec{b}$ for some scalars x and y.
- If the position vectors of the points A and B be $\vec{a}$ and $\vec{b}$ then, the position vectors of the points dividing the line AB in the ratio $m : n$ internally and externally are $\dfrac{m\vec{b}+n\vec{a}}{m+n}$ and $\dfrac{m\vec{b}-n\vec{a}}{m-n}$, respectively.
- If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then $|\vec{r}|=\sqrt{x^2+y^2+z^2}$
- Given vectors $x_1\vec{a}+y_1\vec{b}+z_1\vec{c},\ x_2\vec{a}+y_2\vec{b}+z_2\vec{c},\ x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$, where $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors, will be coplanar if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$
- $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$
- $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$

Scalar Product or Dot Product

- $\vec{a}\cdot\vec{b}=|a|\cdot|b|\cos\theta$, where $0\le\theta\le\pi$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$
- If $\vec{a}$ and $\vec{b}$ are the non-zero vectors, then $\vec{a}\cdot\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}$
- $\cos\theta=\left|\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}\right|$ where θ is the acute angle made by $\vec{a}$ with $\vec{b}$
- Projection of $\vec{b}$ along $\vec{a}=\dfrac{\vec{b}\,.\,\vec{a}}{|\vec{a}|}$
- Component of a vector $\vec{r}$ in the direction of $\vec{a}$ and perpendicular to $\vec{a}$ are $\left(\dfrac{\vec{r}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{r}-\left\{\dfrac{(\vec{r}.\vec{a})}{|\vec{a}|^2}\right\}\vec{a}$ respectively.
- $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$ and $\hat{i}.\hat{j}=\hat{j}.\hat{i}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=\hat{i}.\hat{k}=0$

Vector Product

- The product of vectors $\vec{a}$ and $\vec{b}$ and is denoted by $\vec{a}\times\vec{b}=(|\vec{a}||\vec{b}|\sin\theta)\hat{n}$
- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
- If $\vec{a}=\vec{b}$ or if $\vec{a}$ is parallel to $\vec{b}$, then $\sin\theta=0$ and so $\vec{a}\times\vec{b}=0$
- Distributive laws: $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ and $(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$
- If $\vec{a}=a_1\hat{j}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

  (*i*) $\vec{a}\times\vec{b}=(a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$

  (*ii*) $\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- If two vectors $\vec{a}$ and $\vec{b}$ are parallel, then θ = 0 or π i.e. sin θ = 0 in both cases.
- Two vectors $\vec{a}$ and $\vec{b}$ are parallel if their corresponding components are proportional.
- Area of the triangle $ABC=\dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

  $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0},\ \hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|}$

Scalar Triple Product

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ *and*

  $\vec{c} = c_1 i + c_2 j + c_3 k,$ then $(\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$
- $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{b}\ \vec{a}]$ but $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$ etc.
- If any two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[\vec{a}\ \vec{b}\ \vec{c}] \quad 0.$
- The position of dot and cross in a scalar triple product can be interchanged. Hence, $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$
- The value of a scalar triple product is zero if two of its vectors are parallel.
- $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0.$
- Volume of the parallelepiped whose coterminous edges are formed by $\vec{a},\vec{b},\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}].$
- Volume of a tetrahedron with three coterminous edges

  $\vec{a},\vec{b},\vec{c} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|.$
- Volume of prism on a triangular base with three coterminous edges $\vec{a},\vec{b},\vec{c} = \frac{1}{2}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|.$
- In particular $\hat{i}.(\hat{j}\times\hat{k}) = 1$

  $[\hat{i}\ \hat{j}\ \hat{k}] = 1$
- $[K\ \vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$
- $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ and $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}]$
- $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$

Vector Triple Product

- If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ are known as vector triple product.
- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$ and $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\,\vec{b} - (\vec{b}.\vec{c})\,\vec{a}$
- $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector in the plane of vectors $\vec{b}$ and $\vec{c}$.
- The vector triple product is not commutative i.e., $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$
- **Lagrange's identity:** $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

  $= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$
- $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

  $= [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a}$

Distance between Lines

(*i*) If two parallel lines are given by

$\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b},$ then distance (d) between them is given by

$$d = \left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

Shortest Distance $= \left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

The two lines directed along $\vec{p}$ and $\vec{q}$ will intersect only if shortest distance = 0.

Reciprocal System of Vectors

- If $\vec{a},\vec{b},\vec{c}$ be any three non-coplanar vectors so that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ then the three vectors $\vec{a}',\vec{b}',\vec{c}'$ defined by the equations $\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a}, \vec{b}, \vec{c}.$
- **Properties of Reciprocal system of vectors:**

  (*i*) $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

  (*ii*) $[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}'\ \vec{b}'\ \vec{c}'] = 1$

  (*iii*) $\vec{i}' = \vec{i}, \vec{j}' = \vec{j}, \vec{k}' = \vec{k}$

  (*iv*) If $\{\vec{a}', \vec{b}', \vec{c}'\}$ is reciprocal system of $\{\vec{a}, \vec{b}, \vec{c}\}$ and $\vec{r}$ is any vector, then

  $\vec{r} = (\vec{r}.\vec{a})\vec{a}' + (\vec{r}.\vec{b})\vec{b}' + (\vec{r}.\vec{c})\vec{c}'$

  $\vec{r} = (\vec{r}.\vec{a}')\vec{a} + (\vec{r}.\vec{b}')\vec{b} + (\vec{r}.\vec{c}')\vec{c}$